A study on some linear evolution equations with time delay

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Résumé : On présente différentes techniques de semi-groupe pour étudier l'équation d'évolution linéaire avec mémoire

$$(VE)\qquad \begin{cases} \dfrac{dx(t)}{dt} = A\,x(t) + \displaystyle\int_0^t B(t-s)\,x(s)\,ds + f(t) \\ x(o) = x_o \in X \end{cases}$$

dans un espace de Banach X.

On généralise l'approche de R.K. Miller de manière à pouvoir traiter une classe plus large d'équations. Les conditions d'existence de ces semi-groupes sont données. L'existence, l'unicité, le bien posé et l'approximation des solutions de l'équation (VE) sont alors déduites des équations différentielles et semi-groupes qui lui sont associées.

Abstract : We present here various semigroup techniques for studying the linear evolution equation with memory

$$(VE)\qquad \begin{cases} \dfrac{dx(t)}{dt} = A\,x(t) + \displaystyle\int_0^t B(t-s)\,x(s)\,ds + f(t) \\ x(o) = x_o \in X \end{cases}$$

in a Banach space X.

A generalization of R:K. Miller's semigroup approach is made so that a borader class of equations can be investigated by his méthod. We determine conditions which ensure the existence of those semigroups. The existence, uniqueness, well-posedness and approximation of the equation (VE) are then derived from the associated differential equations and semigroups.

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1. INTRODUCTION

In this paper, we shall be concerned with the integrodifferential equation

$$\text{(VE)}\qquad \begin{cases} x'(t) = Ax(t) + \int_o^t B(t-s)\, x(s)\, ds + f(t), & t \geq o \\ x(o) = x_o \in X \end{cases}$$

in a Banach space X. Throughout this paper we always assume that A is the infinitesimal generator of C_o semigroup on X and the Hille-Yosida-Phillips conditions

$$||R^n(\lambda\ ;\ A)|| \leq M/(\mathrm{Re}\lambda - \omega)^n, \qquad n \geq 1$$

are satisfied for the resolvent $R(\lambda\ ;\ A)$ for some $\omega \geq o$. Also we assume that f is an element of a Banach space $\mathcal{F}$ of X-valued functions which are defined for $t \geq o$, and that $B(t)$ is a linear operator on $D(A)$ for each $t \geq o$ such that $B(.)\, x \in \mathcal{F}$ for each fixed $x \in D(A)$.

Equation (VE) appears, e.g., in the modelling of heat conduction problems in materials with memory, where $A = \Delta$ is the Laplacian and the kernel $B(t)$ is basically of the form $k(t)A$ with $k \in L^1(o,\infty)$. Linear partial differential integral equations of Volterra type associated with heat conduction problems have been studied by Miller [7], see also the references therein.

In this paper, we are mostly interested in the problems of existence, uniqueness and continuity of solutions with respect to x_o and f. We are also concerned with approximating the solutions $x(t)$ of (VE) by solutions $x_n(t)$ of

$$\text{(VE)}^n\qquad \begin{cases} x'_n(t) = A_n x_n(t) + \int_o^t B_n(t-s)\, x_n(s)\, ds + f(t), & t \geq o \\ x_n(o) = x_o \end{cases}$$

given that $B_n \to B$ and $A_n \to A$ in some sense.

Questions concerning existence uniqueness and well-posedness of solutions of linear Volterra integrodifferential equations in a Banach space have been examined by Miller [6], Miller and Wheeler [8], Chen and Grimmer [1], etc. In the nonlinear case, Crandall and Nohel [2]. Related work concerning integral equations appears in Grimmer and Miller [3], [4].

The approach we are following is similar to that in Miller [6], where he studied (VE) by means of the differential equation

$$(DE)' \quad \begin{cases} \dfrac{d}{dt}\begin{bmatrix} x(t) \\ F(t,.) \end{bmatrix} = C\begin{bmatrix} x(t) \\ F(t,.) \end{bmatrix} \equiv \begin{bmatrix} A & \delta_o \\ B(.) & D_s \end{bmatrix}\begin{bmatrix} x(t) \\ F(t,.) \end{bmatrix}, \quad t \geq o \\ \begin{bmatrix} x(o) \\ F(o,.) \end{bmatrix} = \begin{bmatrix} x_o \\ f \end{bmatrix} \in X \times \mathfrak{F} \end{cases}$$

which is a Cauchy problem on $X \times \mathfrak{F}$. Here $\mathfrak{F} = BU(\mathbb{R}^+ ; X)$ is the space of bounded uniformly continuous functions from $\mathbb{R}^+ = [o,\infty]$ into X, δ_o is the Dirac delta function and D_s is the differentiation operator on $\mathfrak{F}$. Miller proved that solution of (DE)' give solutions to (VE) ([6, Theorem 3.5]). The choice of $\mathfrak{F}$ = BU is necessary ([6, §2]) since $\mathfrak{F}$ must at least contain those bounded uniformly continuous X-valued functions. The proof of his theorem will not go through if the function space $\mathfrak{F}$ is chosen differently, e.g., say $\mathfrak{F} \equiv B^2(\mathbb{R}^+ ; X)$ (the space of Bochner square integrable functions on $\mathbb{R}^+$).

In §2, we first generalize Miller's scheme so that (VE) can be studied for a broader class of function spaces $\mathfrak{F}$. We show that, under appropriate assumptions on B, we have the equivalence of (VE) with a new (DE).

In §3, we compute the resolvent operator $R(\lambda ; c)$ and discuss some properties of the spectrum of C.

The main theorems of this paper are given is §4-6.

In §4, we study the case when $\mathcal{Y} = BU(\mathbb{R}^+ ; X)$. We are particularly interested in the case when $B(t)$ is of the form $k(t)A$.

Existence, uniqueness and well-posedness are proved for such kernels B by using perturbation and decomposition techniques for infinitesimal generators. We also point out an error in a recent paper by J. Zabezyk.

In §5, we study the case $\mathcal{Y} = B^2(\mathbb{R}^+ ; X)$. On this $\mathcal{Y}$ the Dirac delta function δ_o is no longer a bounded linear operator, so it becomes more difficult to derive existence theorems for the associated semigroups. We have proved an existence theorem under some assumptions on the Kernel B, by Lumer-Phillips' theorem.

We study approximations by Trotter's theory in §6.

2. THE EQUIVALENCE BETWEEN A VOLTERRA INTEGRODIFFERENTIAL EQUATION (VE) AND AN ASSOCIATED DIFFERENTIAL EQUATION (DE)

We consider the following differential equation

$$\text{(DE)} \quad \begin{cases} \dfrac{d}{dt} z = Cz \\ z(o) = z_o \in D(C)\ E\ X \times X \times \mathcal{Y} \end{cases}$$

which is a Cauchy problem in the Banach space $X \times X \times \mathcal{Y}$. Here

$$z = \begin{bmatrix} w \\ x \\ y \end{bmatrix} \in X \times X \times \mathcal{Y} \quad \text{with} \quad ||z||^2 \equiv ||w||_X^2 + ||x||_X^2 + ||y||_{\mathcal{Y}}^2$$

$$(2.1) \qquad C = \begin{bmatrix} o & A_o & o \\ o & A & \delta_o \\ o & B & D_s \end{bmatrix}$$

and B is the linear transformation given by $(Bx)(s) = B(s)x$, A_o is a closed operator in X with domain $D(A_o) \supseteq D(A)$ and with resolvent $R(\lambda\ ;\ A_o) = (\lambda I - A_o)^{-1}$. In our treatment, A_o is usually a multiple of either A or perhaps some positive fractional power of A if it exists. D_s is the differentiation operator on $\mathcal{Y}$ defined by $D_s f = f'$ on a domain $D(D_s) \subseteq \mathcal{Y}$ where $f \in D(D_s)$ implies

$$f(s) = \alpha + \int_o^s e(u)du$$

for some $e \in \mathcal{Y}$ and D_s generates the translation semigroup $T(t)$ on $\mathcal{Y}$ given by $T(t)\ f(s) = f(t+s)$. The domain of C, D(C), is $X \times D(A) \times D(D_s)$. It is routine to verify that C is a closed operator on $X \times X \times \mathcal{Y}$.

We first give some definitions and notations.

Definition 2.1. By a solution $z(t)$ of (DE) satisfying an initial condition $z(o) = z_o$ we mean a function $z : \mathbb{R}^+ \to D(C)$ with z, z' and Cz continuous and $z'(t) = Cz(t)$ for all $t \in \mathbb{R}^+$.

Definition 2.2. A solution $x(t)$ of (VE) satisfying $x(o) = x_o$ is a function $x : \mathbb{R}^+ \to D(A)$ such that x, x' and Ax are continuous and (VE) is satisfied for all $t \in \mathbb{R}^+$.

Definition 2.3. The equation (DE) is uniformly well-posed if for each $z_o \in D(C)$ the initial value problem $z(o) = z_o$ has a unique solution $z(t,z_o)$ and for any $T > o$ there is a $K > o$ such that : $||z(t,z_o)|| \le K||z_o||$ for all $z_o \in D(C)$ for all $t \in [o,T]$.

Definition 2.4. The equation (VE) is uniformly well-posed is for each pair (x_o,f) with $(o,x_o,f) \in D(C)$ there is a unique solution $x(t,x_o,f)$ of (VE) and for any $T > o$ there is an $M > o$ such that $||x(t,x_o,f)||_X \le M(||x_o||_X + ||f||_{\mathcal{Y}})$ for all $t \in [o,T]$.

Notations : From now on, $T(t)$ always denote the translation semigroup. We use h_s to denote the translated function $T(s)h$, i.e., $h_s(u) = h(s+u)$. $\mathcal{L}_\lambda$ denotes the Laplace transform, i.e., $\mathcal{L}_\lambda h = \int_0^\infty e^{-\lambda s} h(s)ds$. We use $*$ to denote the transpose of an element in $X \times X \times \mathcal{Y}$.

In order to obtain an equivalence relation between solutions of (VE) and those of (DE), we require the following assumptions dealing with B :

(H1) $B(.)x(t)$ is continuous as a function of t on $\mathbb{R}^+$ into $\mathcal{Y}$ whenever $x(t)$ and $A_0x(t)$ are continuous on $\mathbb{R}^+$ into X.
In addition , $B(.)R(\lambda,A_0)$ is a bounded operator from X into $\mathcal{Y}$.

(H2) $B(s)x$ is in $D(D_s)x$ for each fixed x in $D(A_0)$.

(H3) $D_sB(s)x(t)$ is locally integrable as a function of t whenever $x(t)$ and $A_0x(t)$ are continuous.

(H4) $A_0x(t)$ is continuous as a function of t whenever $Ax(t)$ is continuous as a function of t.

An example of a function $B(t)$ which may satisfy (H1)-(H3) is $B(t) = a(t)A_0$, where $a(t)$ is a scalar valued function. Consider $\mathcal{Y} = B^2(\mathbb{R}^+; X)$.

If $a \in L^2(\mathbb{R}^+)$, (H1) is satisfied. If a is absolutely continuous with $a' \in L^2(0,\infty)$, then $B(s)x=a(s)A_0x$ is in $D(D_s)$ for each fixed $x \in D(A_0)$ and $a'(s)A_0x(t)$ is a continuous function of t into $\mathcal{Y}$ when $A_0x(t)$ is continuous. So (H2) and (H3) are satisfied. (H4) is satisfied if $A = -A_0^2$ and A_0 is invertible, or if $A = A_0$.

The following theorem generalizes [6, Theorem 3.5].

<u>Theorem 2.5</u>. Assume (H1) is valid. If $z = (w,x,y)^*$ is a solution of (DE), then $x(t)$ is a solution of (VE) with $f(t) = y(o)(t)$ and $x_o = x(o)$. Conversely, if $f \in D(D_s)$ and if (H1)-(H4) are valid and if $x(t)$ is a solution of (VE), then $(w,x,y)^*$ is a solution of (DE) with $w(t) = w_o + \int_o^t A_o x(s)ds$

and $y(t)(s) = f\ (t+s) + \int_o^t B(t-\tau+s)\ x(\tau)\ d\tau$

<u>Proof</u> : First assume that (H1) is valid and $z = (w,x,y)^*$ is a solution of (DE). Then w, w', x, x', y and y' are all continuous as functions of t from $\mathbb{R}^+$ into either X or $\mathcal{Y}$. As the equation

$$y'(t) = D_s y(t) + B(s)\ x(t)\ ;\ y(o) = y_o, \quad t \geq o$$

has a solution, it follows from [9, p. 110] or [5, p. 488] that the solution is given by

$$y(t) = T(t)\ y_o + \int_o^t T(t-\tau)\ B(.)\ x(\tau)\ d\tau$$

where $T(t)$ is the semigroup generated by D_s, i.e., $T(t)$ is the translation semigroup. Hence if $y(o) = f$, we see that

$$y(t)(s) = f(t+s) + \int_o^t B(t-\tau+s)\ x(\tau)\ d\tau$$

As $y \in D(D_s)$, $y(t,.)$ is absolutely continuous and so

$$y(t)(o) = f(t) + \int_o^t B(t-\tau)\ x(\tau)\ d\tau$$

is continuous in t by (H1). Now $x'(t) = Ax(t) + y(t)(o)$ so $Ax(t)$ is continuous. Therefore

$$x'(t) = Ax(t) + \int_o^t B(t-\tau)\ x\ (\tau)\ d\tau + f(t)$$

is a solution of (VE).

Conversely, of (H1)-(H4) are valid and $x(t)$ is a solution of (VE) with $f \in D(D_s)$, then $x(t)$ and $Ax(t)$ are continuous so that $B(.)x(t)$, which is in $D(D_s)$ for each t, is continuous in t and $D_sB(s)x(t)$ is locally integrable as a function of t. Thus, the equation

$$y'(t) = D_s y(t) + B(s)x(t)$$

has as its solution [9, p. 112]

$$y(t)(s) = f(t+s) + \int_0^t B(t-\tau+s)\, x(\tau)\, d\tau$$

and in particular,

$$y(t)(o) = f(t) + \int_0^t B(t-\tau)\, x(\tau)\, d\tau$$

so

$$x'(t) = Ax(t) + \delta_0 y(t)$$

Thus, if $w(t) = w_0 + \int_0^t A_0 x(s)ds$, $z \equiv (w,x,y)^*$ is a solution of $z'=Cz$ with $z(o)=(w_0,x_0,y_0)$. Q.E.D.

We note that, under the assumptions (H1)-(H4), if the solution of (DE) are unique, then the solutions of (VE) with $(o,x_0,f) \in D(C)$ are unique when they exist. Similarly, if the solutions of (VE) are unique for $(o,x_0,f) \in D(C)$ then the solutions of (DE) must also be unique. It follows that if C generates a C_0 semigroup, then (VE) is uniformly well-posed. Thus we will be concerned with conditions which ensure that C generates a C_0 semigroup in the subsequent sections.

3. RESOLVENTS AND SPECTRUM OF THE OPERATOR C

Let C be given as in (2.1). The following computation of the resolvent in a generalization of [6, Theorem 4.1].

Theorem 3.1 For any λ with $\mathrm{Re}\lambda > o$, $R(\lambda;C)$ exists if and if only $R(x, A+\mathcal{L}_\lambda B)$ exists. If $R(\lambda;C)$ eixists, it is given by

$$\text{(3.1)} \qquad R(\lambda;C) = \begin{bmatrix} \frac{1}{\lambda} & \frac{1}{\lambda} A_o R(\lambda;A+\mathcal{L}_\lambda B) & \frac{1}{\lambda} A_o R(\lambda;A+\mathcal{L}_\lambda B)\mathcal{L}_\lambda \\ o & R(\lambda;A+\mathcal{L}_\lambda B) & R(\lambda;A+\mathcal{L}_\lambda B)\mathcal{L}_\lambda \\ o & R(\lambda;D_s)BR(\lambda;A+\mathcal{L}_\lambda B) & R(\lambda;D_s)[I+BR(\lambda;A+\mathcal{L}_\lambda B)]\mathcal{L}_\lambda \end{bmatrix}$$

Proof : Assume $R(\lambda; C)$ exists. Then for any $(f,g,h) \in X \times X \times \mathcal{F}$, the equation

$$\begin{bmatrix} \lambda I & -A_o & o \\ o & \lambda I - A & -\delta_o \\ o & -B & \lambda I - D_s \end{bmatrix} \begin{bmatrix} w \\ x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \\ h \end{bmatrix}$$

is always solvable with solution (w,x,y). Thus

$$\begin{cases} \lambda w - A_o x = f \\ (\lambda I - A)x - \delta_o y = g \\ -Bx + (\lambda I - D_s)y = h \end{cases}$$

so

$$y = R(\lambda;D_s)(h+Bx)$$

i.e. $$y(s) = \int_o^\infty e^{-\lambda u}\, T(u)\, [h(s) + B(s)\, x]\, du$$

$$= \int_o^\infty e^{-\lambda u}\, [h_u(s) + B_u(s)x]\, du$$

$$= \int_o^\infty e^{-\lambda u}\, [h_s(u) + B_s(u)x]\, du$$

Therefore

$$(3.2) \qquad \delta_o y = \int_0^\infty e^{-\lambda u} \; [h(u) + B(u)x] \; du = \mathcal{L}_\lambda \; (h + Bx)$$

Also,
$$X = R(\lambda;A) \; [g + \delta_o y]$$
$$= R(\lambda;A) \; [g + \mathcal{L}_\lambda (h+Bx)] \qquad \text{(from (3.2.))}$$

so
$$[I - R(\lambda;A)\mathcal{L}_\lambda B] \; x = R(\lambda;A)(g+ \mathcal{L}_\lambda h)$$

the above relation is always invertible with solution

$$x = [I - R(\lambda;A)\mathcal{L}_\lambda B]^{-1} \; R(\lambda;A)(g+\mathcal{L}_\lambda h)$$

But
$$[I - R(\lambda;A)\mathcal{L}_\lambda B\,]^{-1} \; R(\lambda;A) = (\lambda I - A - \mathcal{L}_\lambda B)^{-1} = R(\lambda;A+ \mathcal{L}_\lambda B)$$

Hence $R(\lambda;A+\mathcal{L}_\lambda B)$ exists.

The first component w is given by

$$w = \frac{1}{\lambda} \; (f + A_o x)$$

$$= \frac{1}{\lambda} \; [f + A_o R(\lambda;A+\mathcal{L}_\lambda B)(g + \mathcal{L}_\lambda h)]$$

We note that $A_o R(\lambda;A+ \mathcal{L}_\lambda B)$ is a bounded operator by the closed graph theorem.

Conversely, if $R(\lambda;A+ \mathcal{L}_\lambda B)$ exists. Because each of the above steps is reversible, $R(\lambda;C)$ must also exist and is given by (3.1). Q.E.D.

We note, in particular, that $R(\lambda;A+\mathcal{L}_\lambda B)$ exists provided that $\mathcal{L}_\lambda B$ is a bounded operator on X and $\mathrm{Re}\lambda > \omega + M \; ||\mathcal{L}_\lambda B||$. Having found $R(\lambda;C)$, one can proceed with matrix multiplications to find the iterated resolvent $R^n(\lambda;C)$. Because $R^n(\lambda;C)$ does not have a simple representation, it is in general very difficult to verify whether the Hille-Yosida-Phillips criterion is satisfied. This makes any attempt impractical to directly prove that C is infinitesimal generator.

From the appearance of C, we know that its spectrum depends on the spectrum of A and D_s as well as the behavior of B. We refer the readers to [10] for the definitions of resolvent and point, continuous and redidue spectrum, which we denote by ρ, $p\sigma$, $c\sigma$ and $r\sigma$, respectively.

Different $\mathcal{F}$s give different spectrum for D_s. For example , if $\mathcal{F} \equiv BU(R^+;X)$, then

$$\begin{aligned}
\rho\,(D_s) &= \{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda > o\} \\
p\sigma(D_s) &= \{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda \leq o\} \\
c\sigma(D_s) &= r\sigma(D_s) = \emptyset
\end{aligned}$$

but if $\mathcal{F} \equiv B^2(R^+;X)$, then

$$\begin{aligned}
\rho(D_s) &= \{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda > o\} \\
p\sigma(D_s) &= \{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda < o\} \\
c\sigma(D_s) &= \{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda = o\} \\
r\sigma(D_s) &= \emptyset
\end{aligned}$$

The spectrum of the operator C can usually be classified by a careful computation.

An important subset of the spectrum of C, called the essential spectrum $e\sigma(C)$, merits special attention. There are many non-equivalent definitions of $e\sigma$. We will use the one given in [5].

The following theorem indicates the invariance of the essential spectrum under the perturbation by B when B and A have certain properties.

Theorem 3.2. Let C_1 denote the operator $\begin{bmatrix} o & A_o & o \\ o & A & \delta_o \\ o & o & D_s \end{bmatrix}$

on $X \times X \times \mathcal{Y}$. Assume that for some λ , A has a compact resolvent $R(\lambda;A)$. If the operator $\mathcal{B}$ defined by $\mathcal{B}\, x \equiv B(.)x$ is a bounded linear operator from X into $\mathcal{Y}$, then for (i) $\mathcal{Y} \equiv BU(\mathbb{R}^+;X)$ or (ii) $\mathcal{Y} \equiv B^2(\mathbb{R}^+;X)$ and X = a Hilbert space, C has the same essential spectrum as C_1.

Proof : We want to show that the bounded operator $\bar{B}$ defined by

$$\bar{B}(w,x,y)^* \equiv (o,o,\mathcal{B}\, x)^* \in X \times X \times \mathcal{Y}$$

is C_1-compact ([5]).

Let $\{(w_n, x_n, y_n)\}$ be a sequence in $D(C_1)$ ($\equiv D(C)$) bounded in $X \times X \times \mathcal{Y}$ such that $\{C_1(w_n, x_n, y_n)^* = (A_o x_n, Ax_n + \delta_o y_n, y'_n)\}$ is bounded. (i) $\mathcal{Y} \equiv BU(\mathbb{R}^+;X)$: this implies that $\{Ax_n\}$ is bounded in X, so $\{(\lambda I-A)x_n\}$ is also bounded in X. Hence

$$x_n = (\lambda I - A)^{-1} (\lambda I - A)x_n$$

has a convergent subsequence, which we still denote by $\{x_n\}$. Thus

$$||\bar{B}(w_n, x_n, y_n) - \bar{B}(w_m, x_m, y_m)||_{X \times X \times \mathcal{Y}}$$

$$= \sup_{t \in R^+} ||B(t)x_n - B(t)x_m||_X \le ||\mathcal{B}||_{\mathcal{L}(X,\mathcal{Y})} ||x_n-x_m||_X \to o \text{ as}$$

$$n, m \to \infty$$

Therefore $\bar{B}$ is C_1-compact.

(ii) $\mathcal{Y} \equiv B^2(R^+;X)$: since both $\{y_n\}$ and $\{y'_n)$ are bounded, so is

$$\frac{I}{2} |\delta_o y_n|_X^2 = | \langle y'_n, y_n \rangle_{\mathcal{Y} \times \mathcal{Y}} |$$

Thus $\{Ax_n\}$ (= $\{(Ax_n + \delta_o y_n) - \delta_o y_n\}$) as a sum of two bounded sequences is also bounded. So is $\{(\lambda I - A)x_n\}$. Hence

$$x_n = (\lambda I - A)^{-1} (\lambda I - A)x_n$$

has a convergent subsequence which we still denote by $\{x_n\}$ as before. Now

$$||\bar{B}(w_n, x_n, y_n) - \bar{B}(w_m, x_m, y_m)||^2_{X \times X \times \mathcal{F}}$$

$$= \int_0^\infty ||B(t)(x_n - x_m)||^2_X \, dt \leq ||\mathcal{B}||^2_{\mathcal{L}(X;\mathcal{F})} \, ||x_n - x_m||^2 \to 0 \text{ as } n,m \to \infty$$

Therefore $\bar{B}$ is also C_1-compact.

Since $C = C_1 + \bar{B}$, by [5, Theorem 5.35] the proof is complete. Q.E.D.

Remark : For any bounded invertible operator, P, $P^{-1}C_1P$ has the same spectrum classification as C_1, as $R(\lambda;P^{-1}C_1P) = P^{-1}R(\lambda;C_1)P$ is true for $\lambda \in \rho(C_1)$. This observation will be useful in the subsequent proof of theorem 4.1.

4. THEOREMS ON EXISTENCE, UNIQUENESS AND CONTINUITY(I) : $\mathcal{F} \equiv BU(R^+;X)$

Throughout this section we assume that $\mathcal{F} \equiv BU(R^+;X)$ and $A_o = A$. For this $\mathcal{F}$, (DE) may also be treated in the simpler setting $X \times \mathcal{F}$ such as in [1], [6]. It is easy to see in the subsequent treatment that any results valid in the setting $X \times \mathcal{F}$ are also valid in $X \times X \times \mathcal{F}$, and vice versa.

Theorem 4.1. Suppose B can be written as $B = FA + K$ where $F : X \to \mathcal{F}$ with range $F \equiv D(D_s)$ and $K : X \to \mathcal{F}$ are bounded linear operators. Then C generates a C_o semigroup on $X \times X \times \mathcal{F}$.

Remark : The above theorem holds for any $\mathcal{F}$ such that δ_o is a bounded operator from $\mathcal{F}$ into X.

Proof : We first note that the operator on $X \times X \times \mathcal{Y}$ given by

$$C_1 = \begin{bmatrix} o & o & o \\ o & A & o \\ o & o & D_s \end{bmatrix}$$

generates a C_o semigroup on $X \times X \times \mathcal{Y}$. The operator on $X \times X \times \mathcal{Y}$ given by

$$P = \begin{bmatrix} I_X & -I_X & o \\ o & I_X & o \\ o & -F & I_{\mathcal{Y}} \end{bmatrix}$$

is invertible with inverse

$$P^{-1} = \begin{bmatrix} I_X & I_X & o \\ o & I_X & o \\ o & F & I_{\mathcal{Y}} \end{bmatrix}$$

and for any $(w, x, y) \in P^{-1}(D(C_2)) = P^{-1}(X \times D(A) \times D(D_s))$,

$$P^{-1}C_2P(w, x, y)^* = P^{-1}C_2(w, x, y, -F x + y)$$

$$= P^{-1}(o, Ax, D_s(-F x + y))$$

$$= (Ax, Ax, FAx + D_s(-Fx+y)) = (Ax, Ax, FAx - D_sFx + D_sy)$$

Since F maps X into $D(D_s)$, we have $P^{-1}(X \times D(A) \times D(D_s)) = X \times D(A) \times D(D_s)$. Also, D_sF is a closed operator from X into $\mathcal{Y}$ with domain X. By the closed graph theorem, D_sF is a bounded operator. Thus

$$P^{-1}C_2P = \begin{bmatrix} o & A & o \\ o & A & o \\ o & FA-D_sF & D_s \end{bmatrix}$$

we must remark that in general the above is not true if F does not map X into $D(D_s)$. Now

$$E = \begin{bmatrix} o & o & o \\ o & o & \delta_o \\ o & D_s F+K & o \end{bmatrix}$$

is a bounded operator, so $P^{-1}C_2P+E = C$ is an infinitesimal generator of a C_o-semigroup [9, P. 50] or [5, P. 497] with $D(C) = X \times D(A) \times D(D_s)$ Q.E.D.

Corollary 4.2. (Hille-Yosida-Phillips' conditions). Let A satisfy the Hille-Yosida-Phillips conditions $||R^n(\lambda;A)|| \leq M(\mathrm{Re}\lambda-\omega)^{-n}$ for some $\omega \geq o$ and let α_1, α_2, α_3 denote

$$\alpha_1 = ||F||_{\mathcal{L}(X,\mathfrak{Y})},\ \alpha_2 = ||D_sF||_{\mathcal{L}(X,\mathfrak{Y})},\ \alpha_3 = ||K||_{\mathcal{L}(X,\mathfrak{Y})}$$

Then the iterated resolvent $R^n(\lambda;C)$ is bounded by

$$||R^n(\lambda;C)||_{\mathcal{L}(X \times X \times \mathfrak{Y})} \leq \frac{M(2+\alpha_1)^2}{[\mathrm{Re}\lambda-\omega-M(2+\alpha_1)^2(1+\alpha_2+\alpha_3)]^n} \qquad \text{all } n \in \mathbb{Z}^+$$

if $\mathrm{Re}\lambda$ is large enough.

Proof. Since $R(\lambda;P^{-1}C_2P) = P^{-1}R(\lambda;C_2)P$, therefore

$$R^n(\lambda;P^{-1}C_2P) = P^{-1}R^n(\lambda;C_2)P$$

and

$$||R^n(\lambda;P^{-1}C_2P)|| \leq ||P^{-1}||\ ||R^n(\lambda;C_2)||\ ||P||$$

$$\leq \frac{M(2+\alpha_1)^2}{[\mathrm{Re}\lambda-\omega]^n} \qquad (\because ||P|| \leq 2+\alpha_1,\ ||P^{-1}|| \leq 2+\alpha_1)$$

Now $C = P^{-1}C_2P + E$

From The proof of [9, Theorem 3.1.1.], one easily sees that

$$||R^n(\lambda;C)|| \leq \frac{M(2+\alpha_1)^2}{[\mathrm{Re}\lambda-\omega-M(2+\alpha_1)^2(1+\alpha_2+\alpha_3)]^n}$$

because $||E||_{\mathcal{L}(X \times X\mathfrak{Y})} \leq 1 + \alpha_2 + \alpha_3$. Q.E.D.

Corollary 4.3. Suppose (H1)-(H4) are valid with $B(t) = a_1(t)A + a_2(t)I$, where a_1, a'_1, a_2 and a'_2 are bounded uniformly continuous scalar functions on R^+. The integral equation (VE) is uniformly well-posed for (x_o,f) with (o,x_o,f) in $D(C)$.

We remark that this corollary is essentially Miller's Theorem 7.3. [6] which was obtained via greatly different techniques. Other results, similar to those in [6, §7] follow in a similar manner.

Acutally, a much more general result follows from Theorem 4.1. As A generates a C_o semi-group, A has a resolvent $R(\lambda,A) = (\lambda I - A)^{-1}$ for λ with $\mathrm{Re}\lambda$ sufficiently large. As in [6, p. 181], Miller noted that if $x(t)$ satisfies (VE), then $\gamma(t)$ defined by $\gamma(t) = x(t)\exp(-\lambda_o t)(\lambda_o > o)$ satisfies the equation

$$(VE)_\lambda \quad \gamma'(t) = (A - \lambda_o I)\,\gamma(t) + \int_o^t \exp(-\lambda_o(t-s))\,B(t-s)\gamma(s)ds + \exp(-\lambda_o t)\,f(t)$$

We may then, without any essential loss of generality, assume

($WLOG_1$) A has a bounded inverse A^{-1}, or

($WLOG_2$) A generates a uniformly bounded semigroup. (By changing the norm on X, we can actually assume that A generates a contraction semigroup).

Corollary 4.4. Under the convention of ($WLOG_1$), assume (H1)-(H4) are valid for $\mathcal{Y} = BU(\mathbb{R}^+;X)$ with $A_o = A$ and assume furthermore

$$||B(t)x|| \le \beta\,(||x|| + ||Ax||), \quad \text{for } x \in D(A), \quad \beta > o$$

Then the integral equation (VE) is uniformly well-posed for (x_o,f) with $(o,x_o,f) \in D(C)$.

Proof : Define $F : X \to D(D_s)$ by $F = BA^{-1}$. Then $B = FA$ and

$$||Fx||_{BU(\mathbb{R}^+;X)} = \sup_{t\in\mathbb{R}^+} ||B(t)A^{-1}x|| \le \beta(||A^{-1}x|| + ||x||)$$

So F is a bounded operator. The proof follows from theorem 4.1. Q.E.D.

The techniques used in Theorem 4.1. were motivated by results obtained recently by Zabczyk [11]. In fact, it appears at first that a slight modification of Zabczyk's theorem 1 part 2 would allow us to obtain our result even if F does not map X into the domain of D_s. This is not the case as Zabczyk's result is not correct. First of all the computations at lines 18 and 19 on [11, p. 525] gives $\mathcal{T}^{-1}\mathcal{A}\mathcal{P}$ rather than $\mathcal{A}\mathcal{P}$ Secondly, we have a counterexample which shows that :

$$\begin{bmatrix} I & F \\ o & I \end{bmatrix} \begin{bmatrix} A & o \\ o & B \end{bmatrix}$$

need not generate a semigroup if F does not map into the domain of A. Consider the operator

$$\begin{bmatrix} D & o \\ o & D \end{bmatrix} \quad , \quad \text{i.e., } A \equiv B \equiv D$$

which generates a semigroup on X x X if D generates a semigroup S(t) on X. Now consider the operator

$$\begin{bmatrix} I & I \\ o & I \end{bmatrix} \begin{bmatrix} D & o \\ o & D \end{bmatrix} = \begin{bmatrix} D & D \\ o & D \end{bmatrix} \equiv \mathcal{A}$$

If Zabczyk's result were correct, $\mathcal{A}$ would generate a C_o semigroup on X x X. Now choose $x_o \in D(D)$ which has the property that $S(t)Dx_o \notin D(D)$ for any $t \geq o$. Then $(o,x_o)^* \in D(\mathcal{A})$and there must be a classical solution of $w' = \mathcal{A}w$, $w = (x,y)^*$ through this point if $\mathcal{A}$ generates a C_o semigroup. This solution must have second coordinate $S(t)x_o$. The first coordinate must satisfy

$$\begin{cases} y' = Dy + DS(t)x_o \\ y(o) = y \end{cases} .$$

However, as shown in Pazy [9, p.111] this problem has no solution for if y where a solution then y(t) must satisfy

$$y(t) = \int_o^t S(t-s)\, DS(s)x_o ds = t\, S(t)\, Dx_o$$

This is impossible, however, as $S(t)\, Dx_o$ differentiable would mean $S(t)\, Dx_o \in D(D)$. Interestingly, the operator D which yields the easiest such example is $D = D_s$ on $X \equiv BU(\mathbb{R}^+ ; R)$. Choosing x_o to be a function with only one derivative in $BU(\mathbb{R}^+ ; R)$, we are clear that translating the function will not smooth the function in general.

We restate a corrected version of Zabczyk's result in the following, which can be proven as above or will follow from his Theorem 1 a(1). All of his subsequent related results must also be modified accordingly.

Theorem 4.5. Let X and Y be Banach spaces and A and B generate C_o semigroups on X and Y respectively. If $F : Y \to D(A)$ then

$$\begin{bmatrix} I & F \\ o & I \end{bmatrix} \begin{bmatrix} A & o \\ o & B \end{bmatrix}$$

Also generates a semigroup.

An additional existence theorem will be given in §6 (Theorem 6.5.').

5. THEOREMS ON EXISTENCE, UNIQUENESS AND CONTINUITY (II) : $\mathcal{F} = B^2(R^+;X)$

If the Dirac delta function is not a bounded operator on $\mathcal{F}$ into X, different techniques must be used. This is the case when $\mathcal{F} = B^2(\mathbb{R}^+ ; X)$ of course. Our results in this direction are not as general because the unbounded operator δ_o must also be dealt with. What we have obtained here are similar to Theorems 3 and 4 of our earlier work [1] for the case $\mathcal{F} = BU(\mathbb{R}^+ ; X)$.

Throughout this section, we will follow the convention $(WLOG_2)$ in §4.

Theorem 5.1. Let X be a real Hilbert space with inner product < , >. Let A be the generator of a contraction semigroup on X and let $\mathcal{F} = B^2(\mathbb{R}^+;X)$ with the usual norm. Suppose $B(.) : X \to \mathcal{F}$ is defined on the domain of A at least and $||B(.)x||^2 \le -M_1 < Ax,x>$ and $||A_o x||_X^2 \le -M^2 < Ax,x >$ for all $x \in D(A)$ for some positive constants M_1, M_2. Assume that $R(\lambda+\alpha;A+\mathcal{L}_\lambda B)$ exists as a bounded operator on X for some $\lambda > o$ and some α with $2\alpha > M_1+M_2$. Then C generates a C_o semigroup $||S(t)||$ with $||S(t)|| \le e^{\alpha t}$.

Proof : Consider the operator

$$\begin{bmatrix} -\alpha I_X & A_o & o \\ o & A-\alpha I_X & \delta_o \\ o & B & D_s-\alpha I_{\mathcal{Y}} \end{bmatrix}$$

on $X \times D(A) \times D(D_s)$. We wish to employ the theorem of Lumer-Phillips [9] for the operator C_α. We first show that C_α is dissipative. Using $<\ ,\ >$ as inner product in each space $X, \mathcal{Y}$ and $X \times X \times \mathcal{Y}$, we see that if $z = (w,x,y) \in D(C_\alpha)$,

$$< C_\alpha z, z > = < -\alpha w + A_o x, w >_X + < (A-\alpha I)x + \delta_o y, x >_X + < B_x + (D_s-\alpha I)y, y >_{\mathcal{Y}}$$

$$= -\alpha (||w||^2 + ||x||^2 + ||y||^2) + <A_o x, w> + <Ax,x> + <\delta_o, y, x> + <Bx,y> + <D_s, y, y>$$

$$\leq -\alpha (||w||^2 + ||x||^2 + ||y||^2) + ||A_o x|| \, ||w|| + < Ax,x> + ||\delta_o y|| \, ||x|| + ||B(.)x||_{\mathcal{Y}} ||y||_{\mathcal{Y}} + <D_s y, y>$$

Now $<D_s y,y> = -||\delta_o y||_X^2 / 2$ and

$$||Bx||_{\mathcal{Y}} \, ||y||_{\mathcal{Y}} \leq (\varepsilon^2 \, ||Bx||_{\mathcal{Y}} + \varepsilon^{-2} \, ||y||^2)/2$$

Similarly ,

$$||A_o x|| \, ||w|| \leq (\varepsilon^2 \, ||A_o x||^2 + \varepsilon^{-2} \, ||w||^2)/2$$

Choosing ε sufficiently small and $\alpha = \max\left((\varepsilon^{-2}+1)/2,\ (M_1+M_2+1)/2\right)$, we will obtain $< C_\alpha z, z > \leq o$. So C_α is a dissipative operator.

Now consider $\lambda I - C_\alpha$. We want to show that the range of $\lambda I - C_\alpha$ is $X \times X \times \mathcal{Y}$ for some $\lambda > o$. But

$$(\lambda I - C_\alpha)^{-1} = ((\lambda+\alpha)I - C^{-1}) = R(\lambda+\alpha;\ C)$$

exists provided that $R(\lambda+\alpha;\ A + \mathcal{L}_\lambda B)$ exists by theorem 3.1.

Therefore C_α generates a contraction semigroup and, hence, C generates a semigroup $S(t)$ with $||S(t)|| \leq e^{\alpha t}$. Q.E.D.

Applying this to our original problem, we have

Corollary 5.2. Let X ,A and $\mathcal{F}$ be as above. Suppose (H1)-(H4) are valid with $||B(t)x||_X^2 \leq - b(t) < Ax,x >$ a.e. $\mathbb{R}^+$ for some $b \in L^1(\mathbb{R}^+)$ and $||A_o x||^2 \leq - M <Ax,x>$ for some $M > o$. Assume $R(\lambda+\alpha;A+\mathcal{L}_\lambda B)$ exists as a bounded operator on X for some $\lambda > o$ and some α with $\alpha > 1/2(M+\int_o^\infty b(t)dt)$. Then (VE) is uniformly well posed for (x_o,f) where $(o,x_o f) \in D(C)$.

6. APPROXIMATIONS

In this section we will be concerned with approximating the solutions of (VE) by those of $(VE)_n$. A result of this kind has been obtained in our earlier work [1, Theorem 5]. Here we will study this problem under the general setting of this paper. Our results are motivated by a close examination of the proofs of Theorems 4.1/and 5/1.

First, we consider the differential equations

$(DE)_n$ $$z'_n = C^n z_n,\quad z_n(o) = z(o) \in X \times X \times \mathcal{F}$$

where

$$C^n \equiv \begin{bmatrix} o & A_n & o \\ o & A_n & \delta_o \\ o & F_n A_n & D_s \end{bmatrix}$$

$$\mathcal{F} \equiv BU(\mathbb{R}^+ ;\ X)$$

We shall assume that A_n generates a C_o semigroup on W and $F_n : X \to D(D_s)$ for all n. Theorem 4.1. implies that each operator C^n generates a C_o semigroup.

Theorem 6.1. Suppose $\{A_n\}$ and A are infinitesimal generators of C_o semigroups $\{S_n(t)\}$ and $\{S(t)\}$ such that $\{A_n\}$ and A are defined on a common domain D(A) and $A_n x \to Ax$ for every $x \in D(A)$. Suppose there are constants $M > o$, $\omega \geq o$ such that $||S_n(t)|| \leq Me^{\omega t}$ and $||S(t)|| \leq Me^{\omega t}$. Suppose further that F_n and F are bounded linear operators mapping X into $D(D_s)$ such that $F_n x \to Fx$ and $D_s F_n x \to D_s F_x$ in $\mathfrak{Z}$ for all $x \in X$. Then with $B \equiv FA$, if $z_n(o)=z(o)=z_o$ is in $D(C) = D(C^n)$, we have $z_n(t) \to z(t)$ as $n \to \infty$ for all $t \geq o$ and the convergence is uniform on bounded t intervals.

Proof : We first note that $C = P^{-1}C_2P+Q$ and $C^n=P_n^{-1}C_n^2P_n+Q_n$ where

$$C_2^n = \begin{bmatrix} o & o & o \\ o & A_n & o \\ o & o & D_s \end{bmatrix}$$

$$P = \begin{bmatrix} I & -I & o \\ o & I & o \\ o & -F & I \end{bmatrix} , \qquad P_n = \begin{bmatrix} I & -I & o \\ o & I & o \\ o & -F_n & I \end{bmatrix}$$

$$Q = \begin{bmatrix} o & o & o \\ o & o & \delta_o \\ o & D_sF & o \end{bmatrix} , \qquad Q_n = \begin{bmatrix} o & o & o \\ o & o & \delta_o \\ o & D_sF_n & o \end{bmatrix}$$

Our assumptions have the effect of making P, P_n, Q and Q_n bounded operators. In fact, by the uniform boundedness principle, $P_n \to P$, $P_n^{-1} \to P^{-1}$ and $Q_n \to Q$ in the uniform operator topology and there is a uniform bound for $||P_n^{-1}||, ||P^{-1}||, ||P_n||, ||P||, ||Q_n||$ and $||Q||$; call it M_1.

We choose λ with $Re\lambda > \omega + MM_1^3$. From [9. p. 50], we see that $\{C_n\}$ and $\{C\}$ generates semigroups $\{S_n(t)\}$ and $\{S(t)\}$ satisfying

$$||S_n(t)|| \le MM_1^2\, e^{(\omega+MM_1^3)t}, \quad ||S(t)|| \le MM_1^2\, e^{(\omega+MM_1^3)t}$$

with the resolvent conditions

$$||R^n(\lambda\ ;\ C_m)|| \le \frac{MM_1^2}{(Re\lambda - \omega - MM_1^3)^n}$$

$$||R^n(\lambda;\ C\ )|| \le \frac{MM_1^2}{(Re\ \lambda - \omega - MM_1^3)^n}$$

Now we want to show that for all $z \in X \times X \times \mathcal{F}$, $R(\lambda;\ C_n)\ z \to R(\lambda;C)z$ as $n \to \infty$. Since $R(\lambda;C_n) = P^{-1}R(\lambda;C_2^n + P_nQ_nP_n^{-1})\ P_n$ and $R(\lambda;C)=P^{-1}R(\lambda;C_2+PQP^{-1})P$, this is equivalent to showing that $R(\lambda;C_2^n+P_nQ_nP_n^{-1})z \to R(\lambda;C_2+PQP^{-1})z$ for all z. Let

$$k = R(\lambda;C_2+PQP^{-1})z\ , \quad k \in D(C)$$

and let $\quad z_n \equiv [\lambda I-(C_2^n+P_nQ_nP^{-1})]\ k$

From the given assumptions, we see immediately that z_n tends to z as $n \to \infty$.

Now

$$R(\lambda;C_2^n+P_nQ_nP_n^{-1})z = R(\lambda;C_2^n+P_nQ_nP_n^{-1})z_n + R(\lambda;C_2^n+P_nQ_nP_n^{-1})(z-z_n)$$

and

$$\lim_{n\to\infty} |R(\lambda;C_2^n+P_nQ_nP_n^{-1})(z-z_n)|$$

$$\le \lim_{n\to\infty} \frac{M}{(Re\lambda-\omega-MM_1^3)} |\ z-z_n| = o$$

Hence $$\lim_{n\to\infty} R(\lambda; C_2^n + P_n Q_n P_n^{-1}) z = \lim_{n\to\infty} R(\lambda; C_2^n + P_n Q_n P_n^{-1}) z_n$$

$$= k = R(\lambda;\ C_2 + PQP^{-1}) z$$

It now follows from Trotter's approximation theorem [9, p. 57] or [5, p. 504] that the proof is complete. Q.E.D.

Corresponding to the case $\mathcal{F} = B^2(\mathbb{R}^+;X)$ and Theorem 5.1, we consider

$$(DE)_n \qquad z'_n = C^n z_n \quad , \quad z_n(o) = z(o)$$

$$C^n = \begin{bmatrix} o & A_o^{(n)} & o \\ o & A_n & \delta_o \\ o & B_n & D_s \end{bmatrix}$$

and obtain the following similar result.

<u>Theorem 6.2</u>. Let X be a real Hilbert space with inner product $< , >$. Let $\mathcal{F} = B^2(\mathbb{R}^+;X)$ be equipped with the usual norm. Assume that all the assumptions of Theorem 5.1. are satisfied for each triple $(A_n, B_n, A_o^{(n)})$ and (A, B, Ao) with the same constants M_1, M_2 and α . Assume furtheremore $D(A_n) = D(A)$, $D(A_o^{(n)}) = D(A_o)$ and for all $x \in D(A)$, $A_n x \to Ax$, $A_o^{(n)} x \to A_o x$ in X and $B_n x \to B_x$ in $\mathcal{F}$. If $z_n(o) = z(o) = z_o$ is in $D(C) = D(C^n)$, we have $z_n(t) \to z(t)$ as $n \to \infty$ for all $t \geq o$ and the convergence is uniform on bounded t intervals.

<u>Proof</u>. Note first that each of the operators C^n generates a semigroup $S_n(t)$ with $\|S_n(t)\| \leq e^{\alpha t}$. We are thus able to argue as in the previous theorem that because $C^n z \to Cz$ for $z \in D(C)$ that $R(\lambda; C^n) z \to R(\lambda; C) z$ for all $z \in X \times X \times \mathcal{F}$. The proof again follows form the Trotter approximation theorem.

Theorems 6.1 and 6.2 have immediate application to our stated objective of obtaining results which ensure that the solution $x_n(t)$ of $(VE)_n$ tends to the solution $x(t)$ of (VE). Corresponding to theorem 6.1. we are able to

obtain the following result.

<u>Theorem 6.3</u>. Let (H1) - (H4) be valid for $\mathcal{F}=BU(R^+,X)$ and $A_o=A$. Suppose A_n and A are the generators of C_o semigroups $\{S_n(t)\}$ and $\{S(t)\}$ respectively and that $||S_n(t)|| \leq Me^{\omega t}$, $||S(t)|| \leq Me^{\omega t}$ for some constants $M>o$, $\omega>o$. Suppose that the operators A_n and A have common domain $D(A)$ and that $A_n x \to Ax$ for every $x \in D(A)$. Also, suppose that $B_n(.)x \to B(.)x$ in $BU(R^+;X)$ for every $x \in D(A)$ and that $||B_n(.)x||_{BU} \leq \beta(||x||+||A_n x||)$, $||B'_n(.)x|| \leq \beta(||x||+||A_n x||)$ for all $x \in D(A)$ for some positive constant β for all n. Then for $(o,x_o,f) \in D(C)$, we have $x_n(t) \overset{n\to\infty}{\to} x(t)$ pointwise in t for all $t \geq o$. The convergence is uniform on bounded t intervals.

<u>Proof</u> : We first argue in a similar fashion as in the proof of Theorem 6.1., we obtain

$$(6.1) \qquad R(\lambda;A_n)y \to R(\lambda;A)y \quad n\to\infty \qquad y \in X, \ \forall\lambda > \omega$$

Now, instead of considering $(VE)_n$ and (VE), we consider $(VE)_{n,\lambda}$ and $(VE)_n$. We make the factorization

$$B_{n,\lambda} \equiv \exp(-\lambda s)B_n = F_n(A_n-\lambda I), \text{ with } F_n \equiv -B_{n,\lambda} R(\lambda;A_n)$$

$$B\lambda \equiv \exp(-\lambda s)B = F(A-\lambda I), \text{ with } F \equiv - B_\lambda R(\lambda;A)$$

For $x \in X$, we have $R(\lambda;A)x \in D(A)$, $R(\lambda;A_n)x \in D(A)$ and

$$(6.2) \qquad ||F_n x - Fx||_{BU} \leq ||B_\lambda R(\lambda;A)x - B_{n,\lambda} R(\lambda;A)x|| + $$

$$+ ||B_{n,\lambda} R(\lambda;A)x-B_{n,\lambda} R(\lambda;A_n)x||$$

The first term on the right of (6.2) vanishes as $n \to\infty$. Consider the second term

$$||B_{n,\lambda} R(\lambda;A)x - B_{n,\lambda} R(\lambda;A_n)x|| \le \beta \, (|| R(\lambda;A)x - R(\lambda;A_n)x|| +$$
$$+ ||A_n R(\lambda;A)x - A_n R(\lambda;A_n)x||$$

Using (6.1), we see the vanishing of the first term on the right as $n \to \infty$. And

$$||A_n R(\lambda;A)x - A_n R(\lambda;A_n)x|| \le ||A_n R(\lambda;A)x - AR(\lambda;A)\ x|| + ||AR(\lambda;A)x - A_n R(\lambda;A_n)x|$$
$$= ||A_n R(\lambda;A)x - AR(\lambda;A)x|| + ||\lambda R(\lambda;A)x - \lambda R(\lambda;A_n)x|| \xrightarrow{n \to \infty} o$$

We thus obtain $F_n\, x \to Fx$ for every $x \in X$.

Similarly, we can prove that $D_s F_n \to D_s Fx \quad \forall x \in X$.

By theorem 6.1, the proof is complete. Q.E.D.

Theorem 6.3. has an interesting application to integrodifferential equations arising in the study of heat conduction in materials with memory. In particular, Miller [7] examined the equation

$$\text{(HVE)} \qquad \theta'(t) = c\,\Delta\theta(t) - a(o)\,\theta(t) + c \int_o^t b(t-\tau)\Delta\theta\,(\tau)\,d\tau -$$
$$- \int_o^t a'(t-\tau)\,\theta(\tau)\,d\tau + f(t)$$
$$\theta(o) = \emptyset_o$$

where $\Delta$is the Laplacian, $c > o$, a' and b are continuously differentiable real valued functions in $L^1(R^+)$. Assumptions made regarding the set Ω on which the Laplacian is considered and on the boundary conditions make $A \equiv c\Delta - a(o)I$ a generator of a C_o semigroup on $L^P(\Omega)$, $1 < p < \infty$. If instead of (HVE) we consider

$$(HVE)_n \qquad \theta'_n(t) = c\Delta\theta_n(t) - a_n(o)\theta_n(t) + c\int_o^t b_n(t-\tau)\Delta\theta_n(\tau)d\tau -$$

$$- \int_o^t a'_n(t-\tau)\theta_n(\tau)d\tau + f(t)$$

$$\theta_n(o) = \emptyset$$

we see that Theorem 6.3. can be immediately applied. If the assumptions $b_n \to b$, $b'_n \to b'$, $a_n \to a'$ and $a''_n \to a''$ in $BU(R^+)$ are satisfied, we see that $\theta_n(t)$ converges to $\theta(t)$ uniformly on bounded intervals. In particular, if $b \equiv b' \equiv a' \equiv a'' = o$, then $\theta_n(t)$ converges to $\theta_o(t)$ which is the solution of

$$(HE) \qquad \theta'_o(t) = c\Delta\theta_o(t) + f(t)$$

$$\theta_o(o) = \emptyset_o$$

We thus conclude that if a and b are small (in the sense of Theorem 6.3), the solution $\theta(t)$ of (HVE) differs only slightly from $\theta_o(t)$ because of the memory term.

The above discussion leads us to consider a related problem. If $x(t,\epsilon)$ is the solution of the equation

$$(VE)_\epsilon \qquad x'(t) = Ax(t) + \epsilon\int_o^t B(t-s)\,x(s)\,ds + f(t)\,, \qquad x(o) = x_o$$

we would like to compare $x(t,\epsilon)$ with $x(t,o)$, the solution of

$$(VE)_o \qquad x'(t) = Ax(t) + f(t) \qquad\qquad , \qquad x(o) = x_o$$

The corresponding differential equations are

$$(DE)_\epsilon \qquad z' = C(\epsilon)z$$

and

$$(DE)_o \qquad z' = C(o)z$$

where

$$C(\epsilon) = \begin{bmatrix} o & A & o \\ o & A & \delta_o \\ o & o & D_s \end{bmatrix} + \epsilon \begin{bmatrix} o & o & o \\ o & o & o \\ o & B & o \end{bmatrix} \text{ and } C(o) = \begin{bmatrix} o & A & o \\ o & A & \delta_o \\ o & o & D_s \end{bmatrix}$$

If we assume that $B = FA$ where $F : X \to \mathcal{Y}$ with Range $F \subseteq D(D_s)$ is a bounded linear operator, we may apply the argument in the proof of Theorem 6.1 to show that for $Re\lambda > \omega_1$

$$||R^n(\lambda;C(\epsilon))|| \leq M_2/(Re\lambda - \omega_1)^n$$

for some constants M_2 and ω_1 which are independent of ε as long as $|\epsilon| \leq 1$. As an immediate consequence of these observations and [5, Theorem 2.19, p. 507] we obtain the following result.

Theorem 6.4. Suppose $B = FA$ where $F : X \to D(D_s)$ is a bounded linear operator. Let $z(t,\epsilon)$ be the solution of $(DE)_\epsilon$ with $z(o) = z_o \in D(C(\epsilon))$. Then

$$z(t,\epsilon) = z(t,o) + \varepsilon z_1(t) + o(\epsilon)$$

In addition, if $x(t,\epsilon)$ is the solution of $(VE)_\varepsilon$ where $(o,x_o,f)^* \in D(C(\epsilon))$, then

$$x(t,\epsilon) = x(t,o) + \varepsilon x_1(t) + o(\epsilon)$$

Theorem 6.1. can be modified in another way so that we may obtain a more general existence theorem for (VE) and (DE). In particular, it removes some restriction that F must map X into $D(D_s)$.

Theorem 6.5. Suppose $\{A_n\}$ and $\{A\}$ are infinitesimal generators of C_o semigroups $\{S_n(t)\}$ and $\{S(t)\}$ such that $\{A_n\}$ and $\{A\}$ have common domain $D(A)$ and $A_n x \to Ax$ for every $x \in D(A)$. Assume there are constants $M > o$, $\omega \geq o$ such that $||S_n(t)|| \leq Me^{\omega t}$ and $||S(t)|| \leq Me^{\omega t}$. Suppose $R(\lambda;A+\mathcal{L}_\lambda B)$ exists for some λ with $Re\lambda>\omega$ and that $\{F_n\}$ and $\{F\}$ are bounded linear operators with F_n mapping X into $D(D_s)$ and $F_n x \to Fx$ in $\mathcal{Y}$ for all $x \in X$. Suppose further that there exists positive constant N so that $||F_n|| + ||D_sF_n|| \leq N$

for all n. If $B = FA$, then C generates a C_o semigroup.

Proof : It follows as in the proof of Theorem 6.1. that

$$||R^n(\lambda;C_k)|| \leq M_2/(\mathrm{Re}\lambda - \omega_1)^n \qquad \mathrm{Re}\lambda > \omega_1$$

for some constants M_2, ω_1 independent of n,k. Also, as the F_n are uniformly bounded, $C^n z \to Cz$ for every $z \in D(C^n) = D(C)$. It follows from Theorem 3.1. that $R(\lambda;C)$ exists and so C must generate a semigroup $\mathcal{S}(t)$ with $||\mathcal{S}(t)|| \leq M_2 e^{\omega_1 t}$ [9, p. 90]. Q.E.D.

A special case of theorem 6.5. applied to (VE) yields the following result.

Theorem 6.6. Suppose $B(t) = a(t)A$ where $a : \mathbb{R}^+ \to \mathbb{R}$ is bounded and uniformly Lipschitzian. If the solution of (VE) are unique when they exists, then (VE) is uniformly well-posed.

Proof : As $B(t) = a(t)A$ with $a(t)$ bounded, $R(\lambda;A+\mathcal{L}_\lambda B)$ exists for all λ with $\mathrm{Re}\lambda$ sufficiently large. Furthermore, as $a(t)$ is uniformly Lipschitzian, it is the uniform limit of a sequence of functions $a_n(t)$ where $a'_n(t)$ is bounded and uniformly continuous. Hence, we take $B_n(t) = a_n(t)A$ and $F_n = a_n I$ in Theorem 6.5. to get that (DE) is uniformly well-posed. As the solutions of (VE) are unique, the results follows. Q.E.D.

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